TEMPERATURE DISTRIBUTION IN MULTILAYERED SEMITRANSPARENT COATINGS

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UDC 536.3:535.34:518.61

The effect of optical thickness on the temperature distribution in a multilayered thermally protective (heat-reflecting) coating is considered. A numerical method of solution is presented for nonlinear problems.

We shall consider the one-dimensional problem of heat conduction in a multilayered heat-reflecting coating with due allowance for the volumetric absorption of the external thermal radiation. Several papers have been devoted to particular cases of the problem [1-3]. In this paper we shall give an analytical solution of the linear steady-state problem for an arbitrary number of layers, and a numerical method of solution for nonlinear problems. We shall allow for the contact resistance between the various layers of the coating, the dependence of the absorption coefficient of the material on the coordinate, and boundary conditions of arbitrary form. We shall give several specific solutions by way of example.

1. Presentation of the Problem. Steady-State Solutions

The temperature distribution in the layers $\xi_{j-1} \leq x \leq \xi_j$ of the coating is described by a succession of quasilinear equations of the parabolic type

$$c_j(x)\rho_j(x)\frac{\partial T_j}{\partial t} = \frac{\partial}{\partial x}\left[\lambda_j(x, T_j)\frac{\partial T_j}{\partial x}\right] + F_j^r + F_j^v, \quad 1 \leqslant j \leqslant n, \quad t \in (t^0, t^1] \tag{1}$$

with boundary conditions

$$F_0\left[t, T(\xi_0), \frac{\partial T}{\partial x}(\xi_0)\right] = 0, \quad F_n\left[t, T(\xi_n), \frac{\partial T}{\partial x}(\xi_n)\right] = 0, \tag{2}$$

discontinuity conditions for the temperature and its derivative with respect to x at the point ξ_j

$$F_j^{(i)}\left[t^0, T(\xi_j \pm 0), \frac{\partial T}{\partial x}(\xi_j \pm 0)\right] = 0, \quad i = 1, 2, \quad 1 \leqslant j \leqslant n-1, \tag{3}$$

and the initial distribution

$$T_j(t^0, x) = T_j^0(x), \quad 1 \leqslant j \leqslant n. \tag{4}$$

The functions F_j^r in Eqs. (1) give the intensity of heat evolution within the volume of the layers resulting from the absorption of thermal radiation, F_j^v gives the intensity of other distributed thermal sources. Neglecting dissipation processes, we may obtain the following differential equation for the radiation flux density q_j^r integrated over all frequencies of the infrared part of the spectrum and averaged with respect to angle

$$dq_j^r = -k_j(x)\, q_j^r dx. \tag{5}$$

Integrating (5) and allowing for the initial condition $q_j^r(\xi_{j-1} + 0) = q_j^{r0}$, $1 \leq j \leq n$, we find an explicit expression for $q_j^r(x)$:

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$$q_j^r(x) = q_j^{r0} \exp\left(-\int_{\xi_{j-1}}^{x} k_j(s)ds\right), \quad 1 \leqslant j \leqslant n. \tag{6}$$

Correspondingly the functions F_j^r take the form

$$F_j^r(x) = -\frac{dq_j^r}{dx} = k_j(x) q_j^{r0} \exp\left(-\int_{\xi_{j-1}}^{x} k_j(s)ds\right). \tag{7}$$

The boundary conditions (2) are not assumed linear and may include terms corresponding to such mechanisms of heat transfer at the surfaces $x = \xi_0$, $x = \xi_n$ as convection, radiation, sublimation, chemical reactions, etc. The discontinuity conditions (3), two at each point ξ_j for $1 \le j \le n-1$, represent the laws of energy and momentum conservation at the points of contact between the layers; these are in general also nonlinear functions of their arguments. These are most frequently used in the form

$$\left(\lambda \frac{\partial T}{\partial x}\right)\Big|_{x=\xi_j-0} = \left(\lambda \frac{\partial T}{\partial x}\right)\Big|_{x=\xi_j+0} = R_j^{-1} T\Big|_{\xi_j-0}^{\xi_j+0}, \tag{8}$$

corresponding to ideally close ($R_j \to 0$) or real ($R_j > 0$) contact between the surfaces. If some of the layers are separated by empty spaces, through which only radiative heat transfer occurs, with an effective emissivity of ε_j, the corresponding discontinuity condition takes the form

$$\left(\lambda \frac{\partial T}{\partial x}\right)\Big|_{\xi_j-0} = \left(\lambda \frac{\partial T}{\partial x}\right)\Big|_{\xi_j+0} = \varepsilon_j \sigma T^4 \Big|_{\xi_j-0}^{\xi_j+0}.$$

There may also be cases in which concentrated heat sources or sinks occur at individual points in the gaps separating the layers of the coating. All these forms of thermal interaction of the layers are contained as particular cases in (3). Subsequently we shall use a dimensionless form of Eqs. (1) and the initial data (4):

$$\frac{\partial u_j}{\partial \tau} = Q_j^{(1)} \frac{\partial^2 u_j}{\partial x^2} + Q_j^{(2)}, \quad Q_j^{(i)} = Q_j^{(i)}\left(\tau, x, u_j, \frac{\partial u_j}{\partial x}\right), \quad i = 1, 2, \tag{1'}$$

$$u_j(\tau^0, x) = u_j^0(x), \quad 1 \leqslant j \leqslant n. \tag{4'}$$

Equations (2) and (3) may also be converted into dimensionless form and united into a single system, which together with (4') may then be considered as the system of boundary conditions of a many-point boundary problem for Eqs. (1') with $(\tau, x) \in [\tau^0, \tau^1] \times [\xi_0, \xi_n]$:

$$f_j^{(i)}\left[\tau, u(\xi_{j-1} \pm 0), \frac{\partial u}{\partial x}(\xi_{j-1} \pm 0), u(\xi_j \pm 0), \frac{\partial u}{\partial x}(\xi_j \pm 0)\right] = 0, \quad i = 1, 2; \ 1 \leqslant j \leqslant n. \tag{3'}$$

In Eqs. (1'), (3'), (4'), and subsequently everywhere else, x is treated as a dimensionless coordinate. We also introduce the dimensionless absorption coefficient K, related to k and the characteristic linear dimension L by the relation $K = Lk$.

Let us consider the steady-state linear problem with the additional assumptions $\lambda_j = \text{const}$, $F_j^\gamma \equiv 0$. In this case Eqs. (1') take the form

$$\frac{d^2 u_j}{dx^2} = q_j K_j(x) \exp\left(-\int_{\xi_{j-1}}^{x} K_j(s)\, ds\right), \quad 1 \leqslant j \leqslant n. \tag{9}$$

It is easy to check that the general solution of each of these equations is given by the expression

$$u_j(x) = \alpha_j (x - \xi_j) + \beta_j + v_j(x). \tag{10}$$

The functions $v_j(x)$ are particular solutions of the inhomogeneous equation

$$v_j(x) = q_j \int_{\xi_{j-1}}^{x} \left[1 - \exp\left(-\int_{\xi_{j-1}}^{s} K_j(u)\, du\right)\right] ds.$$

Thus the problem of integrating Eq. (1') and of satisfying the boundary conditions (3') are separate in the present case. The solution of the first problem is given by Eqs. (10). The arbitrary constants α_j and β_j in (10) are found from the system of boundary conditions (3'). In the case in which the discontinuity conditions have the very simple form (8) and the boundary conditions (2) are linear, the problem of finding the coefficients α_j, β_j reduces to a solution of the system of 2n linear equations:

$$a_0\alpha_1 + b_0\beta_1 + c_0 = 0 \ (F_0 = 0);$$

$$\alpha_j + \frac{dv_j}{dx}(\xi_j) = (\lambda_{j+1}/\lambda_j)\,\alpha_{j+1} = (L/\lambda_j R_j)\,[\beta_{j+1} - \alpha_j(\xi_j - \xi_{j-1}) - \beta_j - v_j(\xi_j)];$$

$$a_n\left[\alpha_n + \frac{dv_n}{dx}(\xi_n)\right] + b_0[\alpha_n(\xi_n - \xi_{n-1}) + \beta_n + v_n(\xi_n)] + c_n = 0,\ 1 \leqslant j \leqslant n-1.$$

In the limiting case of an optically infinitely thick outer layer with $K_1 \to \infty$, the zone of absorption of the external radiation moves to the surface $x = \xi_0$. We integrate Eq. (9) for j = 1 between ξ_0 and $\xi_0 + \varepsilon$:

$$\frac{du_1}{dx}(\xi_0 + \varepsilon) = \frac{du_1}{dx}(\xi_0) + q_1 - q_1 \exp(-\bar{K}_1\varepsilon).$$

If we successively take $\bar{K}_1$ to ∞ and ε to zero, we obtain a new boundary condition for $x = \xi_0$

$$\frac{du_1}{dx}(\xi_0 + 0) = Q_F + q_1.$$

Here Q_F denotes all the "truly" surface dimensionless thermal fluxes introduced from outside, while q_1 is the flux of radiation absorbed in an infinitely thin layer close to the surface $x = \xi_0$. In this limiting case Eqs. (9) take an extremely simple form

$$\frac{d^2u_j}{dx^2} = 0,\ 1 \leqslant j \leqslant n.$$

2. Transient Problems. Examples

The transient problem of Eq. (1') with conditions (3') and (4') is solved numerically by means of the implicit scheme:

$$Q_j^{(1)}\frac{d^2u_j}{dx^2} + Q_j^{(2)} = (1+k)\frac{u_j - v_j}{h_v} - k\frac{u_j - w_j}{h_w} \equiv D_\tau(u_j),\ k = \frac{h_v}{h_v - h_w}. \tag{11}$$

The right-hand side of this equation constitutes the derivative $\partial u_j/\partial\tau(\tau, x)$ expressed in terms of the v_j and w_j distributions respectively associated with the instants of time $\tau - h_V$ and $\tau - h_W$. At the first step in the integration with respect to τ the parameter k in (11) is formally taken as zero. Since the implicit scheme is stable for any ratio of the time step to the space step, integration with respect to τ may be carried out with an independent choice of steps, satisfying any preassigned accuracy. The nonlinear boundary problem for Eqs. (11) with boundary conditions (3') is solved at every fixed instant of time τ by a certain modification of the span method with iterations. The formulas of the corresponding algorithm were considered in detail in [4] and may be applied to the present case without any changes. The steady-state distributions are obtained, either as the natural result of the settling of the transient solution, or else as the set of values obtained at the first time step for which it is sufficient to treat the step h_W in Eqs. (11) as extremely large ($h_W(D_\tau(u_j) \to 0)$). As a criterion for judging how closely the approximation $u_j^{(k)}$ of the k-th iteration approaches the exact solution of the boundary problem, we may take the functional of the mean-square deviation (mismatch) with respect to Eqs. (11) and conditions (3'):

$$\delta_k = \sum_{j=1}^{n} |f_j^{(1)}| + \frac{1}{(\xi_j - \xi_{j-1})}\left(\int_{\xi_{j-1}}^{\xi_j}\left[Q_j^{(1)}\frac{\partial^2 u_j^{(k)}}{\partial x^2} + Q_j^{(2)} - D_\tau(u_j^{(k)})\right]^2 dx\right)^{\frac{1}{2}} + |f_j^{(2)}|.$$

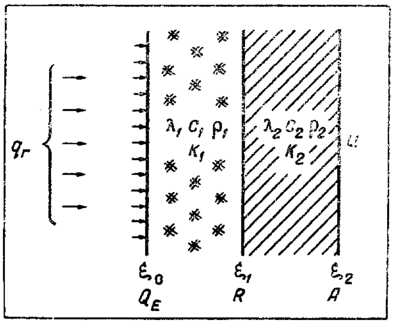

Fig. 1. Two-layer coating–substrate system.

As a solution to the boundary problem (11), (3') we take that $\{u_j^{(k)}\}_1^n$ for which the inequality $\delta_k \le \varepsilon_1$ is satisfied, where ε_1 is some preassigned positive number, accepted as the permissible error in the iterative process of solving (11), (3'). If we have already found the solution $\{u_j\}$ for the instants of time τ_W and τ_V ($\tau_W < \tau_V$), then the solution at the instant $\tau(\tau = \tau_V = h_V)$ will be constructed as set out above. Let us use P to denote the corresponding operator

$$\{u_j(\tau, x)\} = P[w, \tau_w, v, \tau_v, \tau].$$

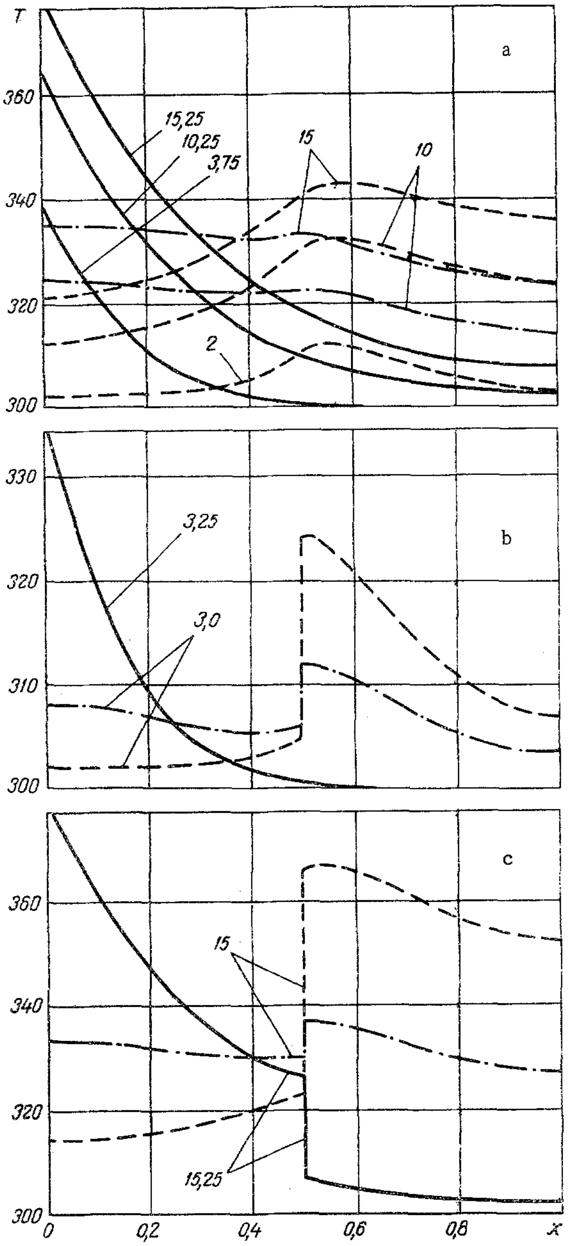

Fig. 2. Temperature distributions: a) for $t = 2$; 3.75; 10; 10.25; 15 and 15.25 sec with $R = 0$; b) for $t = 3$ and 3.25 sec with $R = 1$; c) for $t = 15$ and 15.25 sec with $R = 1$.

The integration step with respect to τ is chosen so as to satisfy the inequalities

$$\left\| P\left[w,\ \tau_w,\ v,\ \tau_v,\ \tau_v + h_v\right] - P\left[v,\ \tau_v,\ P\left[w,\ \tau_w,\ v,\ \tau_v,\ \tau_v + \frac{h_v}{2}\right],\ \tau_v + \frac{h_v}{2},\ \tau_v + h_v\right]\right\| \leqslant \varepsilon_2. \tag{12}$$

Here ε_2 is any prespecified number, taken as the error in the time-integration of Eq. (1'), subject to the conditions (3').

Let us consider some example of the numerical solution of the transient problem for a two-layer system (Fig. 1). The outer layer of this system plays the part of a semitransparent coating, while the inner layer is the substrate, the temperature of which is to be maintained within a specified range of values over a certain interval of time. In order to determine the specific influence of the optical thickness of the layers on the temperature distribution, we take the thermophysical characteristics of the layers and the absorption coefficients as constant. We write Eqs. (1') and (3') in the form

$$\frac{\partial u}{\partial \tau} = Q_1 \frac{\partial^2 u}{\partial x^2} + Q_2,\quad x \in (0,\ x_1) \cup (x_1,\ 1), \tag{13}$$

$$\frac{\partial u}{\partial x}(0) = -Q_E,\ (\lambda_1/\lambda_2)\frac{\partial u}{\partial x}(x_1 - 0) = \frac{\partial u}{\partial x}(x_1 + 0),$$

$$R\frac{\partial u}{\partial x}(x_1 - 0) = u(x_1 + 0) - u(x_1 - 0),$$

$$\frac{\partial u}{\partial x}(1) = A_i(u_i - u). \tag{14}$$

The coefficients of Eq. (13) are determined by the relations

$$0 \leqslant x \leqslant x_1:\ Q_1 = \lambda_1 t_* / c_1\rho_1 L^2,$$

$$Q_2 = (q_r t_* K_1 / c_1\rho_1 T_* L)\exp(-K_1 x),$$

$$x_1 < x \leqslant 1:\ Q_1 = \lambda_2 t_* / c_2\rho_2 L^2,$$

$$Q_2 = (q_r t_* K_2 / c_2\rho_2 T_* L)\exp[(K_2 - K_1)x_1]\exp(-K_2 x).$$

The initial distribution of $u(\tau^0, x)$ was taken as uniform in all the cases considered: $u(\tau^0, x) \equiv u^0$. The thermophysical characteristics of the layers were specified on the basis of the properties of silicon carbide (λ_1, c_1, ρ_1) and titanium (λ_2, c_2, ρ_2) [5]. In view of the absence of reliable data regarding the optical properties of the media in the infrared region, the dimensionless absorption coefficients were specified arbitrarily $K_1 = 0.5$; 2; $K_2 = 10$. The remaining quantities in the dimensionless functions Q_1, Q_2 and the coefficients of the boundary conditions (14) were $t_* = 10^3$ sec, $L = 2$ cm, $x_1 = 1$ cm, $T_* = 10^3$ °K, $u^0 = u_i = 0.3$, $A_i = 10^{-2}$, $Q_E = 0.2$ [6]. We considered two forms of interaction between the layers: ideal contact ($R = 0$) and real contact ($R = 1$). For comparison the problem was solved by each of these versions for the following cases:

a) surface absorption of a thermal flux $Q_E = 0.2$ in the absence of radiation ($\varkappa = q_r L/\lambda_1 T_* = 0$), which is equivalent to the forcing out of the absorption zone to the surface $x = \xi_0(Q_E = 0,\ \varkappa = 0.2,\ K_{11} \to \infty)$;

b) volumetric absorption of an external flux of radiation of the same intensity in the absence of convective heat transfer at the boundary $x = \xi_0(Q_E = 0,\ \varkappa = 0.2,\ K_1 = 0.5;\ 2)$.

In the numerical solution the errors ε_1 and ε_2 were taken as equal to 0.001; the norm $\|\cdot\|$ in (12) coincided with the norm taken in space $C[0, x_1] \cap C[x_1, 1]$. For R = 0 the temperature distributions referred to several instants of time are shown in Fig. 2a. The continuous lines relate to case a), the broken and dotted and dashed lines to case b), for $K_1 = 0.5$ and $K_1 = 2$, respectively. The existence of a contact resistance between the layers leads to characteristic discontinuities in the solutions at the point ξ_1. The corresponding curves for R = 1 are shown in Fig. 2b, c.

3. Discussion

We see from Fig. 2a that the temperature distribution in the layers for $k_1 < \infty$ differs greatly from that expected with purely convective heating, and from the point of view of the thermal protection of the inner substrate surface x = 1 the coating under consideration must be regarded as quite unsuitable. It may appear that the absolute range of variation of the temperature is relatively narrow for all the curves and that for high intensities Q_E the difference in the temperatures at x = 1 becomes (relatively) less appreciable. However, this is not in fact so. If we consider the problem of the pulsed heating of the system during the time interval Δt with a fixed total quantity of heat $Q = Q_E \Delta t$ supplied in the pulse, we find that as $\Delta t \to 0$ the intensity $Q_E \to \infty$. In this limiting case, by virtue of the inertia of heat transfer by conduction, the surface temperature is given by the equation

$$\frac{du}{d\tau} = Q_E - \bar{\varepsilon} u^4,\ Q_E = \begin{cases} Q/\Delta t, & 0 \leqslant \tau \leqslant \Delta t, \\ 0, & \tau > \Delta t. \end{cases}$$

The depth of penetration of the high temperatures into the layers tends to zero as $\Delta t \to 0$. For $\tau > \Delta t$ the outer surface continues to cool by radiation; at high temperatures this process is also much less inertial than the linear heating of the layers by conduction. Thus in the case $k_1 \to \infty$ for short Δt the greater part of the heat will be returned to the medium in the form of radiation and will not fall on the substrate. If, however, the coating layer has a considerable transparency, for example, $k_1 = 0.5$, then for an arbitrarily short Δt the greater part of the heat will be absorbed by the opaque layer $x_1 < x \leq 1$, and the aim of thermally insulating the substrate will not be achieved. The presence of a contact resistance between the layers introduces additional undesirable effects. We see from Fig. 2b, c that for $k_1 < \infty$ (especially for $k_1 = 0.5$) the substrate layer serves as a "trap" for the external radiation q_r. The discontinuity in the temperature value at the point differs not only in magnitude but even in sign. The thermal insulation of the layer $x_1 < x \leq 1$, calculated on the assumption of purely convective heating of the layers for short periods of action (continuous curve in Fig. 2b) may prove unsuitable for $k_1 < \infty$, not only because the substrate is able to heat up over these periods, but also mainly because, at the moment of interest, it has already suffered rupture, owing to the difference in the thermal expansion coefficients of the coating and substrate materials. It was pointed out in [7] that materials such as borides and carbides might be considered as the best refractories at the temperatures of hypersonic flight and entry into the Earth's atmosphere if it were not for their low resistance to thermal shock. The above-mentioned influence of the optical thickness of the coating and the contact resistance between the layers gives a more specific and accurate idea of the reasons underlying this behavior of the coatings.

In conclusion, we note that, in the general setting of (1'), (3'), (4'), the boundary conditions (3') are regarded as arbitrary, and the problem under consideration may be essentially solved quite independently of the outer ($x < \xi_0$) and inner ($x > \xi_n$) problems, whatever these may be. The arbitrary nature of the function $k_j(x)$ in Eq. (5) suggests the possibility of choosing the radiative-optical properties of the system in such a way as to secure some particular form of optimization, for example, minimizing the temperature jump at the point ξ_1 (Fig. 2b, c).

NOTATION

c_j, ρ_j, λ_j	are, respectively, the specific heat, density, and thermal conductivity of the material of the j-th layer;
k_j	is the absorption coefficient of the j-th layer with respect to infrared radiation;
R_j	is the contact thermal resistance at the interface of the j-th and (j−1)-th layers;
q_r	is the density of the external thermal radiation;
t_*, T_*, L	are, respectively, the characteristic time, temperature, and linear dimension of the two-layer system (coating–substrate);
Q_E	is the density of the convective thermal flux at the surface $x = \xi_0$.

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